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Weakly-Compatible and Property (E.A.) in Non-Archimedean Fuzzy Metric-Space

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***Abstract*— In this paper we prove a common fixed point theorem in Non-Archimedean fuzzy metric space using the concept of property (E.A.) and weak compatibility of pair of self map.**

***Keywords* — Non-Archimedean Fuzzy Metric Pace, Property (E.A.), Weak-Compatible.**
Mathematical Subject Classification: 54H25, 47H10.

I. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh [7]. It was developed extensively by many authors and used in various fields. To use this concept in topology and analysis several researcher have defined fuzzy metric space in various ways. George and Veeramani [2] modified the concept of fuzzy metric space introduced by O. Kramosil and Michalek [9]. M.Grebiec [8] has proved fixed point results for fuzzy metric space. S.Sessa [13] defined generalization of commutativity which called weak commmutativity. G.Jungck [6] introduced more generalized commutativity so called compatibility. The concepts of weak compatibility in fuzzy metric space are given by B.Singh and S.Jain [3]. In 1975, V. I. Istratescu [14] first studied the non-Archimedean menger-spaces. They presented some basic topological preliminaries of non Archimedean fuzzy metric space. D.Mihet [4, 5] introduced the concept of Non-Archimedean fuzzy metric space. In this paper we prove common fixed point theorem in Non-Archimedean fuzzy metric space using the concept of property (E.A.) and weak compatibility of pair of self maps.

II. PRELIMINARIES:

Definition 2.1[11] A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$ for all $a \in [0, 1]$,
(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$,

For each $a, b, c, d \in [0, 1]$
Two typical examples of continuous t-norm are
$a * b = ab$ and $a * b = \min(a, b)$.

Definition 2.2[11] The 3-tuple $(X, M, *)$ is called a non-Archimedean fuzzy metric space .If X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:

For all x, y, z $\in$ X and s, t > 0,
(i) $M(x, y, 0) = 0$,
(ii) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \le M(x, z, \max\{t, s\})$
Or equivalently $M(x, y, t) * M(y, z, t) \le M(x, z, t)$
(v) $M(x, y, .): [0, \infty) \to [0,1]$ is left continuous.

Definition 2.3[12] Let $(X, M, *)$ be a non-Archimedean fuzzy metric space:
(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ denoted by

$\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

(ii) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if

$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for all $t > 0$, $p>0$.

(iii) A non-Archimedean fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.4[12] A non-Archimedean fuzzy metric space $(X, M, *)$ is said to be of type (C) g if there exists a $g \in \Omega$ such that

$$g(M(x,y,t)) \le \{g(M(x,z,t)) + g(M(z,y,t))\}$$

for all $x,y,z \in X$ and $t \ge 0$, where $\Omega = \{g : g : [0,1] \to [0,\infty)$ is continuous, strictly decreasing $\{g(1) = 0$ and $g(0) < \infty\}$

Definition 2.5[12] A non-Archimedean fuzzy metric space $(X, M, *)$ is said to be of type (D) g if there exists a $g \in \Omega$ such that $g(*(s,t)) \le g(s)+g(t)$ for all $s,t \in [0,1]$

Definition 2.6[11] Let A and B be mappings from N.A. FM-space $(X, M, *)$ in to itself. The mappings A and B are said to be compatible if

$$\lim_{n\to\infty} g(M(ABx_n, BAx_n, t)) = 0$$

For all t>0 whenever $\{x_n\}$ is a sequence in X
Such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$$

For some $z \in X$.

Definition 2.7[11] A pair of maps A and B is called weakly compatible pair if they commute at coincidence points

i.e., $Ax = Bx$
If and only if $ABx = BAx$.

Definition 2.8[1] The pair (A, S) of an non-Archimedean fuzzy metric space is said to be property (E.A.) if there $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} g(M(Ax_n,u,t) = \lim_{n\to\infty} g(M(Sx_n,u,t)= 0$

For some $u\in X$.

Lemma 2.1 Let (X, M, $*$) be a non-Archimedean fuzzy metric space if there exists $k \in (0, 1)$

Such that $M(x, y, kt) > M(x, y, t)$ for all $x, y \in X$

Then $x = y$

III. Main Result:

Let f and g be two weak compatible self map of a non Archimedean fuzzy metric space (X, M, $*$) satisfy the property (E.A.) and

(i) $fx\subset gx$

(ii) $g\{M(fx,fy,kt)\} \le g\{M(gx,gy,t)\}$

$g\{M(fx,ffx,t)\} \le \varphi$ (max

$$\begin{Bmatrix} g\{M(ggx,gfy,t)\} \\ \sup_{t1+t2}=\frac{2}{k}t\max\{g(M(fx,gx,t_1),g(M(gx,f^2x,t_2))\} \\ \sup_{t3+t4}=\frac{2}{k}t\min\{g(M(f^2x,gfx,t_3),g(M(fx,gfx,t_4))\} \end{Bmatrix}$$

Whenever $fx\neq f^2x$ for all $x, y\in X$, $t>0$ and for some $1\le k<2$. If the range of f and g is a complete subspace of X then f and g have a common fixed point where φ is a continuous function

$fx: [0, 1]\to[0,1]$

Such that $\varphi(t) > t$.

Proof. Since f and g satisfy the property (E.A) there exists a sequence $\{x_n\}$ in X such that fx_n, $gxn \to z$

As $n\to\infty$ for some $z\in X$.Since $z\in fx$ and $fx\subset gx$ therefore there exists some points u in X

Such that $z = gu$

Where $gx_n \to z$

As $n\to\infty$

$fu\neq gu$.

Then

$g\{M(fx_n,fu,kt)\} \le g\{M(gx_n,gu,t)$

Taking limit as $n\to\infty$

we get $g\{M(gu,fu,kt)\} \le g\{M(gu,gu,t)=0$

$M(gu, fu, kt) \ge 1$

Therefore by lemma 2.1 we get

$fu = gu$

Since f and g are weakly compatible so

$fgu = gfu$

And therefore

$fgu = ffu= gfu = ggu.$

If $ffu\neq fu$

Then by contractive condition.

We get

$g\{M(fu,ffu,t_0)\} \le \varphi$ (max

And

$= \varphi$ (max

$$\begin{Bmatrix} g\{M(ffu,ffu,t_0)\} \\ \max\{g(M(fu,fu,\in),g(M(gu,f^2u,\frac{2}{k}t_0-\in)\} \\ \min\{g(M(f^2u,gfu,\in),g(M(fu,ffu,\frac{2}{k}t_0-\in)\} \end{Bmatrix}$$

$\forall\ \varepsilon \in (0,\frac{2}{k}t_0)$ as $\in\to 0$ it follows

$g\{M(fu,ffu,t_0) \le \varphi\ \{g(M(fu,ffu,\frac{2}{k}t_0-\in)\}$

$< g(M(fu,ffu,\frac{2}{k}t_0)\}$

Which is a contradiction

$fu = ffu$

$fu = ffu = fgu = gfu = ggu.$

Hence fu is a common fixed point of f and g.

The case where fx is a complete subspace of X is similar to the above since $fx\subset g(x)$ this complete the proof of the theorem

Remark 3.1: In the next theorem we will show that if we take non- compatible maps in place of property we can show in addition that the mapping are discontinuous at the common fixed point.

Theorem 3.1: Let f and g be two non compatible weakly compatible self mapping of a non Archimedean fuzzy metric space (X, M, $*$)

Such that $fx\subset g(x)$

$g\{M(fx,fy,kt)\} \le g\{M(gx,gy,t)\}$

$g\{M(fx,ffx,t)\} \le \varphi$ (max

$$\begin{Bmatrix} g\{M(ggx,gfx,t)\} \\ \sup_{t1+t2}=\frac{2}{k}t\max\{g(M(fx,gx,t_1),g(M(gx,f^2x,t_2))\} \\ \sup_{t3+t4}=\frac{2}{k}t\min\{g(M(f^2x,gfx,t_3),g(M(fx,gfx,t_4))\} \end{Bmatrix}$$

Whenever $fx\neq f^2x$ for all $x, y \in X$, $t>0$ and for some $1\le k<2$.If the range of f and g is a complete subspace of X then f and g have a common fixed point and the fixed point is the point of discontinuity.

Proof: Since f and g are non compatible maps there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$

For some z in X but either

$\lim_{n\to\infty} g\{M(fgx_n,gfx_n,t)\} \neq 0$

Or the limit does not exists

Since $z\in fx$ and $fx \subset gx$.

There exists some point's u in X

Such that $z = gu$

Where $gx_n \to z$ as $n\to\infty$

We claim that $fu = gu$

Suppose that $fu\neq gu$

Then $g\{M(fx_n, fu, kt)\} \leq g\{M(gx_n, gu, t)\}$

Taking limit as $n \to \infty$ we get

$$g\{M(gu,fu,kt)\} \leq g\{M(gu,gu,t)\} = 0$$

$$g\{M(gu,fu,kt)\} \leq 1$$

Hence $fu = gu$

Since f and g are weakly compatible so

$$fgu = gfu$$

And therefore

$$fgu = ffu = gfu = ggu$$

If

$$ffu \neq fu$$

Then by the contractive condition

$g\{M(fu.ffu,t_0)\} \leq \varphi$ (max

$= \varphi$ (max

$$\left\{\begin{array}{l} g\{M(ffu, ffu, t_0)\} \\ \max\{g(M(fu, fu, \in), g(M(gu, f^2u, \frac{2}{k}t_0 - \in)\} \\ \min\{g(M(f^2u, gfu, \in), g(M(fu, ffu, \frac{2}{k}t_0 - \in)\} \end{array}\right\}$$

$\forall\ \varepsilon \in (0, \frac{2}{k}t_0)$ as $\in \to 0$

It follows $g\{M(fu,ffu,t_0)\} \leq \varphi\ \{g(M(fu,ffu, \frac{2}{k}t_0 - \in)\}$

$$< (fu, ffu, \frac{2}{k}t_0)\}$$

Which is a contradiction

$$fu = ffu$$

$$fu = ffu = fgu = gfu = ggu.$$

Hence fu is a common fixed point of f and g.

Hence fu is a common fixed point of f and the case where fx is a complete subspace of X is similar to the above since $fx \subset gx$ we show that f and g are discontinuous at the common fixed point z = fu = gu. If possible suppose f is continuous then consider the sequence $\{x_n\}$ of (2.1).

We get $\lim_{n\to\infty} ffx_n = fz = z$

Since f and g are weakly compatible so

$$ffu = gfu$$

So $fz = gz$

In $ffx_n = gfx_n$

Taking limit as $n \to \infty$

We get $fz = \lim_{n\to\infty} gfx_n$

or $z = \lim_{n\to\infty} gfx_n$

This is turn yield $\lim_{n\to\infty} g\{M(fgx_n, gfx_n, t)\} = 0$

This contradicts the fact that

$$\lim_{n\to\infty} g\{M(fgx_n, gfx_n, t)\} \neq 0$$

Hence f is discontinuous at the fixed point similarly we can prove point g is discontinues at the fixed point.

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